

### Fourier Series Equations: Practice Problems

**Q1.** Consider the following periodic signals:

- (i)  $x(t) = \sin(4t) + \cos(8t) + 7 + \cos(16t)$
- (ii)  $x(t) = \cos^2(t)$
- (iii)  $x(t) = \cos(t) + \sin(2t) + \cos(3t - \pi/3)$

- (a) Find the trigonometric Fourier series coefficients for each of the above signals.
- (b) Find the compact Fourier series coefficients for each of the above signals.
- (c) Find the complex exponential Fourier series coefficients for each of the above signals.

#### **Q1. Solution.**

(i) Using trigonometric Fourier series we can write the periodic  $x(t)$  as an infinite sum of sinusoids  $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$ . In other words,  $x(t) = \frac{a_0}{2} + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \dots + b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t) + \dots$ . Since the given signal is already written in this form (sum of sinusoids), we can read the coefficients simply *by inspection* (there is no need to perform any integration). All we need is the fundamental frequency of  $x(t)$ , which is  $\omega_0 = 4$  rad/s, since the fundamental frequency of  $x(t)$  is the greatest common divisor of the frequencies of the added sinusoids. Hence,

$$\begin{aligned} x(t) &= \sin(4t) + \cos(8t) + 7 + \cos(16t) \\ &= 7 + \sin(\omega_0 t) + \cos(2\omega_0 t) + \cos(4\omega_0 t) \end{aligned}$$

And the trigonometric Fourier series coefficients are:

$$a_0/2 = 7,$$

$$a_2 = 1,$$

$$a_4 = 1,$$

$$\text{otherwise } a_n = 0$$

$$b_1 = 1,$$

$$\text{otherwise } b_n = 0$$

Similarly, in compact Fourier series,  $x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) = \frac{c_0}{2} + c_1 \cos(\omega_0 t - \theta_1) + c_2 \cos(2\omega_0 t - \theta_2) + c_3 \cos(3\omega_0 t - \theta_3) + \dots$ . Hence, by inspection and noticing that  $\sin(\omega_0 t) = \cos(\omega_0 t - \pi/2)$ , the compact Fourier series coefficients are:

$$\begin{aligned} c_0/2 &= 7, \\ c_1 &= 1, \theta_1 = \pi/2, \\ c_2 &= 1, \theta_2 = 0, \\ c_4 &= 1, \theta_4 = 0, \\ \text{otherwise } c_n &= 0, \theta_n = 0 \end{aligned}$$

In complex exponential Fourier series, we have  $x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t} = \alpha_0 e^{j0} + \alpha_1 e^{j\omega_0 t} + \alpha_{-1} e^{-j\omega_0 t} + \alpha_2 e^{j2\omega_0 t} + \alpha_{-2} e^{-j2\omega_0 t} + \dots$ . Hence,

$$\begin{aligned} x(t) &= 7 + \sin(\omega_0 t) + \cos(2\omega_0 t) + \cos(4\omega_0 t) \\ x(t) &= 7e^{j0} + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + \frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} + \frac{e^{j4\omega_0 t} + e^{-j4\omega_0 t}}{2} \end{aligned}$$

And the complex exponential Fourier series coefficients are:

$$\begin{aligned} \alpha_0 &= 7, \\ \alpha_1 &= -0.5j, \\ \alpha_{-1} &= 0.5j, \\ \alpha_2 &= 0.5, \\ \alpha_{-2} &= 0.5, \\ \alpha_4 &= 0.5, \\ \alpha_{-4} &= 0.5, \\ \text{otherwise } \alpha_n &= 0 \end{aligned}$$

**(ii)** Using the well-known trigonometric identity

$$x(t) = \cos^2(t) = \frac{1}{2} + \frac{1}{2} \cos(2t) = \frac{1}{2} + \frac{1}{2} \cos(\omega_0 t)$$

The fundamental frequency is  $\omega_0 = 2$  rad/s. Also, using Euler's identity,

$$x(t) = \frac{1}{2} + \frac{1}{2} \cos(\omega_0 t) = \frac{1}{2} e^{j0} + \frac{1}{2} \times \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Trigonometric Fourier series coefficients:

$$a_0/2 = 1/2,$$

$$a_1 = 1/2,$$

otherwise  $a_n = 0$

$$\text{all } b_n = 0$$

Compact Fourier series coefficients:

$$c_0/2 = 1/2,$$

$$c_1 = 1/2, \theta_1 = 0,$$

otherwise  $c_n = 0, \theta_n = 0$

Complex Fourier series coefficients:

$$\alpha_0 = 1/2,$$

$$\alpha_1 = 0.25,$$

$$\alpha_{-1} = 0.25,$$

otherwise  $\alpha_n = 0$

**(iii)** The fundamental frequency of  $x(t)$  is  $\omega_0 = 1$  rad/s (the greatest common divisor (GCD))

$$\begin{aligned} x(t) &= \cos(t) + \sin(2t) + \cos\left(3t - \frac{\pi}{3}\right) \\ &= \cos(\omega_0 t) + \sin(2\omega_0 t) + \cos\left(3\omega_0 t - \frac{\pi}{3}\right) \\ &= \cos(\omega_0 t) + \sin(2\omega_0 t) + [A \cos(3\omega_0 t) + B \sin(3\omega_0 t)] \end{aligned}$$

where,

$$A = 1 \times \cos\left(\frac{\pi}{3}\right) = 1/2$$

$$B = 1 \times \sin\left(\frac{\pi}{3}\right) = \sqrt{3}/2$$

Trigonometric Fourier series coefficients:

$$a_0/2 = 0,$$

$$a_1 = 1,$$

$$a_3 = 1/2,$$

otherwise  $a_n = 0$

$$b_2 = 1,$$

$$b_3 = \sqrt{3}/2,$$

otherwise  $b_n = 0$

Compact Fourier series coefficients

(notice, for example, that  $\sin(2\omega_0 t) = \cos(2\omega_0 t - \pi/2)$ ):

$$c_0/2 = 0,$$

$$c_1 = 1, \theta_1 = 0,$$

$$c_2 = 1, \theta_2 = \pi/2,$$

$$c_3 = 1, \theta_3 = \pi/3,$$

$$\text{otherwise } c_n = 0, \theta_n = 0$$

$$\begin{aligned} x(t) &= \cos(\omega_0 t) + \sin(2\omega_0 t) + \cos\left(3\omega_0 t - \frac{\pi}{3}\right) \\ &= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + \frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} + \frac{e^{j\left(3\omega_0 t - \frac{\pi}{3}\right)} + e^{-j\left(3\omega_0 t - \frac{\pi}{3}\right)}}{2} \end{aligned}$$

Complex Fourier series coefficients:

$$\alpha_0 = 0,$$

$$\alpha_1 = 0.5,$$

$$\alpha_{-1} = 0.5,$$

$$\alpha_2 = -0.5j,$$

$$\alpha_{-2} = 0.5j,$$

$$\alpha_3 = 0.5e^{-j\pi/3},$$

$$\alpha_{-3} = 0.5e^{j\pi/3},$$

$$\text{otherwise } \alpha_n = 0$$

**Q2.** If the two halves of one period of a periodic signal are identical in shape except that the second half-cycle is the negative of the first, the periodic signal is said to have *half-wave symmetry*. In other words, a periodic signal  $x(t)$  with period  $T_0$  satisfies half-wave symmetry if,

$$x\left(t \pm \frac{T_0}{2}\right) = -x(t)$$

In this case, show that all the *even*-numbered harmonics vanish, i.e.,

$$a_n = 0, \quad \forall n = \text{even}$$

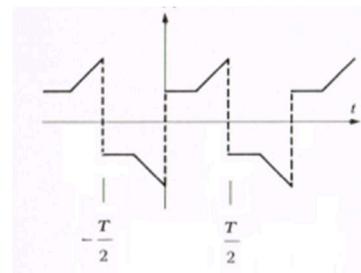
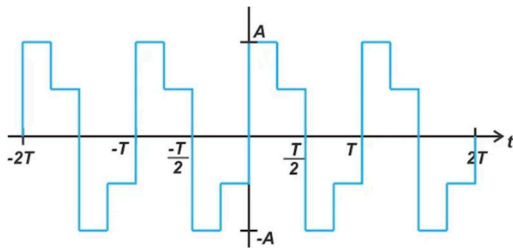
$$b_n = 0, \quad \forall n = \text{even}$$

while the *odd*-numbered harmonic trigonometric coefficients are given by

$$a_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \sin(n\omega_0 t) dt$$

Below are some examples of signals with half-wave symmetry,



**Q2. Solution.**

$$a_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(n\omega_0 t) dt = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

$$a_n = \frac{2}{T_0} \int_{-T_0/2}^0 x(t) \cos(n\omega_0 t) dt + \frac{2}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

Changing variables  $\zeta = t + \frac{T_0}{2}$  or  $t = \zeta - \frac{T_0}{2}$  (and  $d\zeta = dt + 0$ ) for the first integral

$$a_n = \frac{2}{T_0} \int_{\zeta=0}^{T_0/2} x\left(\zeta - \frac{T_0}{2}\right) \cos\left(n\omega_0\left(\zeta - \frac{T_0}{2}\right)\right) d\zeta + \frac{2}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

$$a_n = \frac{2}{T_0} \int_0^{T_0/2} -x(\zeta) \cos(n\omega_0 \zeta - n\pi) d\zeta + \frac{2}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

$$a_n = \frac{2}{T_0} \int_0^{T_0/2} -x(\zeta) (-1)^n \cos(n\omega_0 \zeta) d\zeta + \frac{2}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

Since

$$\cos(n\omega_0 t - n\pi) = \cos(n\pi) \cos(n\omega_0 t) = (-1)^n \cos(n\omega_0 t)$$

For  $n$  even

$$a_n = \frac{2}{T_0} \int_0^{T_0/2} -x(\zeta)(1) \cos(n\omega_0 \zeta) d\zeta + \frac{2}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt = 0$$

For  $n$  odd

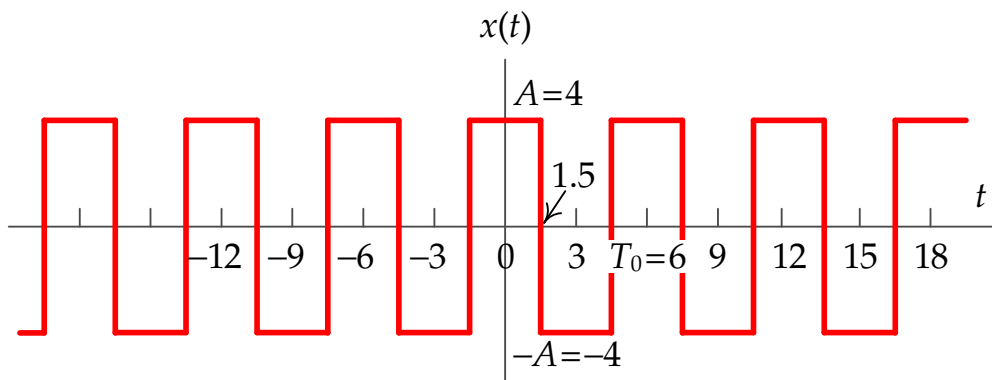
$$\begin{aligned} a_n &= \frac{2}{T_0} \int_0^{T_0/2} -x(\zeta)(-1) \cos(n\omega_0 \zeta) d\zeta + \frac{2}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt \\ &= \frac{4}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt \end{aligned}$$

A similar derivation can be performed for  $b_n$  to get:

$$b_n = 0, \quad \forall n = \text{even}$$

$$b_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \sin(n\omega_0 t) dt, \quad \forall n = \text{odd}$$

**Q3.** Consider the following periodic signal:



- Write a mathematical expression for this signal.
- Find the exponential Fourier series coefficients for this signal.
- Find the trigonometric Fourier series coefficients for this signal.
- Find the compact Fourier series coefficients for this signal.

**Q3. Solution.**

$$x(t) = \text{rep}_6 \left\{ 8 \text{rect} \left( \frac{t}{3} \right) \right\} - 4$$

$$a_0 = 0$$

$$a_n = \frac{8 \times 3}{6} \text{sinc} \left( \frac{n \times 3}{6} \right) = 4 \text{sinc} \left( \frac{n}{2} \right), \quad n \neq 0$$

$$a_0 = 0$$

$$a_n = \frac{2 \times 8 \times 3}{6} \text{sinc} \left( \frac{n \times 3}{6} \right) = 8 \text{sinc} \left( \frac{n}{2} \right)$$

$$b_n = 0$$

$$c_n = \left| \frac{2 \times 8 \times 3}{6} \text{sinc} \left( \frac{n \times 3}{6} \right) \right| = \left| 8 \text{sinc} \left( \frac{n}{2} \right) \right|$$

$$\theta_n = 0^\circ \text{ or } 180^\circ$$